Use of dual numbers in kinematical analysis of spatial mechanisms. Part I: principle of the method

F C Ciornei[1], S Alaci[1], R D Pentiuc[1] and I Doroftei[2]
[1]"Stefan cel Mare" University, Suceava, Romania
[2]"Gheorghe Asachi" Technical University, Iasi, Romania

florina.ciornei@usm.ro

Abstract. The general methodology of solving a problem of kinematical analysis of a spatial mechanism is presented. The method is based on the system proposed by Hartenberg and Denavit, concerning the notations of the coordinate frames attached to the elements of the mechanism and on the closure matrix equation of a kinematical chain. Carrying out the closure equation for the case of spherical mechanisms and then applying the transfer principle of Kotelnikov, the equations which allow for finding the unknown parameters of the kinematical chain are obtained. Starting from the typical form of closure equation based on homogenous operators Hartenberg-Denavit, the closure matrix equation is obtained in dual format. By identifying two special matrices it is shown that the coefficient of the dual part from the closure equation is in fact a sum in which each term is attained from the real part of the closure equation by performing simple operations involving the special matrices, that reduce the calculus volume.

1. Introduction

The dual numbers were introduced by Clifford [1] related to the research conducted in the study of the properties of quaternions. The dual numbers occurred in abstract mathematical researches but after a short time they proved their usefulness in applied sciences, and the work of Ball [2] should be mentioned. In the field of theory of mechanisms Dimentberg [3] and Kotelnikov [4] are essential references. In the domain of theory of mechanisms, Yang [5] obtained important results applying the dual quaternions for the kinematical study of spatial mechanisms; thus, he obtained the analytical relations for the displacements from the pairs of the RCCC mechanism. The dual numbers have a structure similar to the complex numbers from elementary algebra, being expressed under the form:

$$\hat{a} = a + \varepsilon a_0 \tag{1}$$

where ε is the dual unit with the property $\varepsilon^2 = 0$ with the immediate consequence $\varepsilon^n = 0, n > 1$. The matrix method proved to be an extremely useful instrument in completing the kinematical analysis of spatial mechanisms. The matrix denoted $\boldsymbol{d}_{12}$ corresponds to the translation of the origin of the frame $''1''$ over the origin of frame $''2''$ and the matrix related to the rotation $\boldsymbol{R}_{12}$ gets the axes of the system $''1''$ parallel to the axes of the system $''2''$. The transformation relation has the general form:

$$\begin{bmatrix} x_1 & y_1 & z_1 \end{bmatrix}^T = \boldsymbol{d}_{12} + \boldsymbol{R}_{12} \begin{bmatrix} x_2 & y_2 & z_2 \end{bmatrix}^T \tag{2}$$

Or, in a concentrated manner:

$$\boldsymbol{x}_1 = \boldsymbol{d}_{12} + \boldsymbol{R}_{12}\boldsymbol{x}_2 \tag{3}$$

The drawback of this expressing style is evident when it is aimed the equivalent expression of some successive transformations, written as a unique transformation. Thus, writing:

$$\boldsymbol{x}_1 = \boldsymbol{d}_{12} + \boldsymbol{R}_{12}\boldsymbol{x}_2 = \boldsymbol{d}_{12} + \boldsymbol{R}_{12}(\boldsymbol{d}_{23} + \boldsymbol{R}_{23}\boldsymbol{x}_3) = \boldsymbol{d}_{13} + \boldsymbol{R}_{13}\boldsymbol{x}_3 \tag{4}$$

it results:

$$\boldsymbol{d}_{13} = \boldsymbol{d}_{12} + \boldsymbol{R}_{12}\boldsymbol{d}_{23}; \qquad \boldsymbol{R}_{13} = \boldsymbol{R}_{12}\boldsymbol{R}_{23} \tag{5}$$

Hartenberg and Denavit [6] proposed the explanation of the displacement of a system using a (4×4) type matrix, including in its structure both the rotation matrix and the displacement vector:

$$\begin{bmatrix} x_1 & y_1 & z_1 & 1 \end{bmatrix}^T = \begin{bmatrix} & \boldsymbol{R}_{12} & & \boldsymbol{d}_{12} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_2 & y_2 & z_2 & 1 \end{bmatrix}^T \tag{6}$$

or, in concentrated form:

$$\boldsymbol{X}_1 = \boldsymbol{T}_{12}\boldsymbol{X}_2 \tag{7}$$

$$\boldsymbol{X}_1 = \begin{bmatrix} x_1 & y_1 & z_1 & 1 \end{bmatrix}^T,\ \boldsymbol{X}_2 = \begin{bmatrix} x_2 & y_2 & z_2 & 1 \end{bmatrix}^T,\boldsymbol{T}_{12} = \begin{bmatrix} & \boldsymbol{R}_{12} & & \boldsymbol{d}_{12} \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{8}$$

The advantage of expressing the transformation relation under the form (5) is clear for the situation of composing multiple displacements:

$$\boldsymbol{X}_1 = \boldsymbol{T}_{12}\boldsymbol{X}_2 = \boldsymbol{T}_{12}(\boldsymbol{T}_{23}\boldsymbol{X}_3) = (\boldsymbol{T}_{12}\boldsymbol{T}_{23})\boldsymbol{X}_3 = \boldsymbol{T}_{13}\boldsymbol{X}_3 \tag{9}$$

The outcome is:

$$\boldsymbol{T}_{13} = \boldsymbol{T}_{12}\boldsymbol{T}_{23} \tag{10}$$

This expression attests the homogenous nature of transformation relations and thus justifying the name “method of homogenous operators” under which the method is known. In the case of a closed contour the equation of closure of the kinematical chain is obtained:

$$\boldsymbol{T}_{12}\boldsymbol{T}_{23}...\boldsymbol{T}_{n-1,n}\boldsymbol{T}_{n,1} = \boldsymbol{I}_4 \tag{11}$$

Hartenberg and Denavit [6] showed that the displacement of the $"k"$ system over the $"k+1"$ system will be obtained as the result of two helical motions: the first motion, performed about Oz_k axis, with the angle θ_k and the translation s_k and the second, about the $Ox_{k,k+1}$ axis, with the angle $\alpha_{k,k+1}$ and the translation $a_{k,k+1}$. The two above displacement can be completely characterized by the dual numbers:

$$\hat{\theta}_k = \theta_k + \varepsilon\, s_k; \qquad \hat{\alpha}_{k,k+1} = \alpha_{k,k+1} + \varepsilon\, a_{k,k+1} \tag{12}$$

2. The closure equation of the contour in dual form

The two dual angles are represented in figure 1 and figure 2. The two displacements, the roto-translation with respect to Oz_k and the roto-translation with respect to $Ox_{k,k+1}$, will be expressed according to the transfer principle of Kotelnikov [4], using the matrices reflecting the rotation about the axes Oz and Ox respectively, where the actual angles should be replaced by dual angles. In order to obtain the explicit expressions of $z(\theta)$ and $x(\alpha)$ the form of the rotation matrix is particularized:

$$\boldsymbol{R}(\boldsymbol{u},\varphi)=\boldsymbol{u}\boldsymbol{u}^T+\left(\boldsymbol{I}_3+\boldsymbol{u}\boldsymbol{u}^T\right)\cos\varphi+\tilde{\boldsymbol{u}}\sin\varphi \tag{13}$$

where $\boldsymbol{I}_3$ is the unit matrix of third order and $\tilde{\boldsymbol{u}}$ is the antisymmetric matrix attached to the $\boldsymbol{u}$ vector.

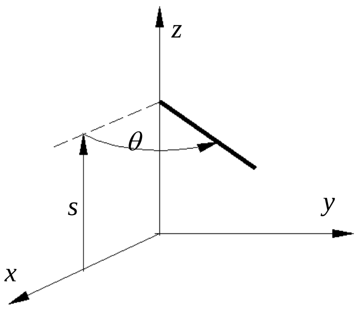

Figure 1. The dual number $\hat{\theta}=\theta+\varepsilon\, s$

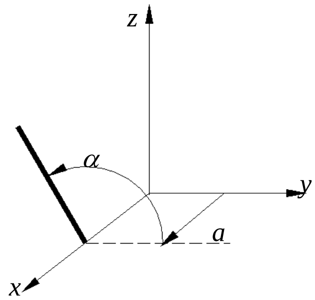

Figure 2. The dual number $\hat{\alpha}=\alpha+\varepsilon\, a$

For $\boldsymbol{u}=\boldsymbol{k}$ and $\varphi=\theta$ and for $\boldsymbol{u}=\boldsymbol{i}$ and $\varphi=\alpha$ it is obtained:

$$z(\theta)=\begin{bmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix};\quad x(\alpha)=\begin{bmatrix}1 & 0 & 0\\ 0 & \cos\alpha & -\sin\alpha\\ 0 & \sin\alpha & \cos\alpha\end{bmatrix} \tag{14}$$

The following dual transformation corresponds to the $\boldsymbol{T}_{k,k+1}$ homogenous transformation:

$$\hat{\boldsymbol{t}}_{k,k+1}=z(\hat{\theta}_k)x(\hat{\alpha}_{k,k+1}) \tag{15}$$

The dual equation that corresponds to the closure equation (11) has the form:

$$\hat{\boldsymbol{t}}_{12}\hat{\boldsymbol{t}}_{23}\ldots\ldots\hat{\boldsymbol{t}}_{n-1,n}\hat{\boldsymbol{t}}_{n,1}=\hat{\boldsymbol{I}}_3 \tag{16}$$

where $\hat{\boldsymbol{I}}_3$ is the dual matrix of third order defined as:

$$\hat{\boldsymbol{I}}_3=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}+\varepsilon\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix} \tag{17}$$

Recalling the definition of equality of two matrices and of two dual numbers respectively, a general equation of form (16) would give a number of $2\cdot(3\times3)=18$ scalar equations. Subsequently it is shown that for the case of a contour from the structure of a mechanism, the relation (16) generates 6 independent equations. In equation (14) the argument θ is replaced by the dual angle $\hat{\theta}$ and it results:

$$z(\hat{\theta})=\boldsymbol{z}(\theta)(\boldsymbol{I}_3+\varepsilon\, s\boldsymbol{Q}_\theta);\quad \boldsymbol{Q}_\theta=\begin{bmatrix}0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{bmatrix} \tag{18}$$

$$\boldsymbol{x}(\alpha)=\boldsymbol{x}(\alpha)(\boldsymbol{I}_3+\varepsilon\, a\boldsymbol{Q}_\alpha);\quad \boldsymbol{Q}_\alpha=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & -1\\ 0 & 1 & 0\end{bmatrix} \tag{19}$$

The closure equation, written in dual form using the notations (14) and (16) take the form:

$$\prod_{k=1}^{n} \boldsymbol{z}(\theta_k)\left[\boldsymbol{I}_3 + \varepsilon s_k \boldsymbol{Q}_\theta\right]\boldsymbol{x}(\alpha_{k,k+1})\left[\boldsymbol{I}_3 + \varepsilon a_{k,k+1}\boldsymbol{Q}_\alpha\right] = \hat{\boldsymbol{I}}_3 \tag{20}$$

The product from the left member of equation (20) is expanded and a polynomial of $2n$ degree with respect to variable ε is obtained, with matrix coefficients. Applying the definition of dual unit, it results that all the terms of second or higher degree must be zero. The equation (20) takes the form:

$$\prod_{k=1}^{n} z(\theta_k)x(\alpha_{k,k+1}) + \varepsilon \sum_{k=1}^{n}(\boldsymbol{P}_{\theta_k} + \boldsymbol{P}_{\alpha_k}) = \hat{\boldsymbol{I}}_3 \tag{21}$$

where $\boldsymbol{P}_{\theta_k}$ and $\boldsymbol{P}_{\alpha_k}$ are matrices obtained by replacing $\boldsymbol{z}(\theta_k)$ by $\boldsymbol{z}_k s_k \boldsymbol{Q}_\theta$, and $\boldsymbol{x}(\alpha_{k,k+1})$ by $\boldsymbol{x}(\alpha_{k,k+1})a_{k,k+1}\boldsymbol{Q}_a$ respectively, in the product $\prod_{k}^{n} z(\theta_k)x(\alpha_{k,k+1})$.

$$\boldsymbol{P}_{\theta k} = \boldsymbol{z}(\theta_1)\boldsymbol{x}(\alpha_{12}).....\boldsymbol{z}(\theta_{k-1})\boldsymbol{x}(\alpha_{k-1,k})\boldsymbol{z}(\theta_k)s_k\boldsymbol{Q}_\theta\boldsymbol{x}(\alpha_{k,k+1})\boldsymbol{z}(\theta_{k+1}).....\boldsymbol{z}(\theta_n)\boldsymbol{x}(\alpha_{n,1}) \tag{22}$$

$$\boldsymbol{P}\alpha_k = \boldsymbol{z}(\theta_1)\boldsymbol{x}(\alpha_{12}).....\boldsymbol{z}(\theta_{k-1})\boldsymbol{x}(\alpha_{k-1,k})\boldsymbol{z}(\theta_k)\boldsymbol{x}(\alpha_{k,k+1})a_{k,k+1}\boldsymbol{Q}_\alpha\boldsymbol{z}(\theta_{k+1}).....\boldsymbol{z}(\theta_n)\boldsymbol{x}(\alpha_{n,1}) \tag{23}$$

The equation (18) is equivalent to the following real equations:

$$\prod_{k=1}^{n} z(\theta_k)x(\alpha_{k,k+1}) = \boldsymbol{I}_3 \tag{24}$$

$$\sum_{k=1}^{n}(\boldsymbol{P}_{\theta_k} + \boldsymbol{P}_{\alpha_k}) = \boldsymbol{O}_3 \tag{25}$$

3. Solving the resultant matrix equations. Observations and discussions

The left member of equation (24) is the product of $2n$ orthogonal matrices, thus being an orthogonal matrix. Considering the physical significance of the elements of an orthogonal matrix, namely that the elements of a $"p"$ column are the projections of the $\boldsymbol{i}_p$ versor on the coordinate axes of the reference frame, it can be concluded that from the nine elements of the rotation matrix only three are independent because the orthonormation conditions must be satisfied:

$$\boldsymbol{i}_p \cdot \boldsymbol{i}_q = \delta_{pq} \tag{26}$$

From here it results the conclusion that only three equations from the nine provided by the matrix equality (24) are independent. Another property of an orthogonal matrix $\boldsymbol{R}$ is:

$$\boldsymbol{R}^{-1} = \boldsymbol{R}^T \tag{27}$$

From equation (24) it is obtained:

$$\boldsymbol{P}_{\theta k} = \boldsymbol{z}(\theta_1)\boldsymbol{x}(\alpha_{12}).....\boldsymbol{z}(\theta_{k-1})\boldsymbol{x}(\alpha_{k-1,k})\boldsymbol{z}(\theta_k)s_k\boldsymbol{Q}_\theta\left[\boldsymbol{z}(\theta_1)\boldsymbol{x}(\alpha_{12}).....\boldsymbol{z}(\theta_{k-1})\boldsymbol{x}(\alpha_{k-1,k})\boldsymbol{z}(\theta_k)\right]^T \tag{28}$$

$$\boldsymbol{P}_{\theta k} = \left[\boldsymbol{x}(\alpha_{k,k+1})\boldsymbol{z}(\theta_{k+1}).....\boldsymbol{z}(\theta_n)\boldsymbol{x}(\alpha_{n,1})\right]^T s_k\boldsymbol{Q}_\theta\boldsymbol{x}(\alpha_{k,k+1})\boldsymbol{z}(\theta_{k+1}).....\boldsymbol{z}(\theta_n)\boldsymbol{x}(\alpha_{n,1}) \tag{29}$$

$$\boldsymbol{P}_{\alpha_k} = \boldsymbol{z}(\theta_1)\boldsymbol{x}(\alpha_{12})....\boldsymbol{z}(\theta_k)\boldsymbol{x}(\alpha_{k,k+1})a_{k,k+1}\boldsymbol{Q}_\alpha\left[\boldsymbol{z}(\theta_1)\boldsymbol{x}(\alpha_{12})....\boldsymbol{z}(\theta_k)\boldsymbol{x}(\alpha_{k,k+1})\right]^T \tag{30}$$

$$\boldsymbol{P}_{\alpha_k} = \left[\boldsymbol{z}(\theta_{k+1}).....\boldsymbol{z}(\theta_n)\boldsymbol{x}(\alpha_{n,1})\right]^T a_{k,k+1}\boldsymbol{Q}_\alpha\boldsymbol{z}(\theta_{k+1}).....\boldsymbol{z}(\theta_n)\boldsymbol{x}(\alpha_{n,1}) \tag{31}$$

Next, the following assertion is necessary: Theorem 1: If $\boldsymbol{Q}$ is an antisymmetric matrix, $\boldsymbol{Q} + \boldsymbol{Q}^T = \boldsymbol{0}$, then for any $\boldsymbol{M}$ matrix, the $\boldsymbol{MQM}^T$ matrix is antisymmetric. Indeed:

$$\boldsymbol{MQM}^T + \left[\boldsymbol{MQM}^T\right]^T = \boldsymbol{MQM}^T + \left[\boldsymbol{M}^T\right]^T\boldsymbol{Q}^T\boldsymbol{M}^T = \boldsymbol{MQM}^T + \boldsymbol{M}(-\boldsymbol{Q})\boldsymbol{M}^T = \boldsymbol{O} \tag{32}$$

The relations (22) and (23), together with equation (24) lead to:

$$\boldsymbol{P}_{\theta k} = \boldsymbol{z}(\theta_1)\boldsymbol{x}(\alpha_{12}).....\boldsymbol{z}(\theta_{k-1})\boldsymbol{x}(\alpha_{k-1,k})\boldsymbol{z}(\theta_k)s_k\boldsymbol{Q}_\theta\left[\boldsymbol{z}(\theta_1)\boldsymbol{x}(\alpha_{12}).....\boldsymbol{z}(\theta_{k-1})\boldsymbol{x}(\alpha_{k-1,k})\boldsymbol{z}(\theta_k)\right]^T \tag{33}$$

$$\boldsymbol{P}_{\alpha_k} = \boldsymbol{z}(\theta_1)\boldsymbol{x}(\alpha_{12}).....\boldsymbol{z}(\theta_k)\boldsymbol{x}(\alpha_{k,k+1})a_{k,k+1}\boldsymbol{Q}_\alpha\left[\boldsymbol{z}(\theta_1)\boldsymbol{x}(\alpha_{12}).....\boldsymbol{z}(\theta_k)\boldsymbol{x}(\alpha_{k,k+1})\right]^T \tag{34}$$

The relations (18) and (19) show that the matrices $\boldsymbol{Q}_\theta$ and $\boldsymbol{Q}_\alpha$ are antisymmetric; based on the relations (33), (34) and Theorem 1, one can state that $\boldsymbol{P}_{\theta_k}$ and $\boldsymbol{P}_{\alpha_k}$ are antisymmetric matrices. As a consequence, the left member of the equation (21) is an antisymmetric matrix and it is completely defined by three scalar parameters; therefore, the matrix equation (21) provides three scalar equations.

The equations (33) and (34) can be simplified following the remark:

$$\boldsymbol{z}(\theta)\boldsymbol{Q}_\theta\boldsymbol{z}(\theta)^T = \boldsymbol{x}(\alpha)\boldsymbol{Q}_\alpha\boldsymbol{x}(\alpha)^T = \boldsymbol{I}_3 \tag{35}$$

The above relations are verified by direct calculus. Based on the relations (35) it is obtained:

$$\boldsymbol{P}_{\theta k} = \boldsymbol{z}(\theta_1)\boldsymbol{x}(\alpha_{12}).....\boldsymbol{z}(\theta_{k-1})\boldsymbol{x}(\alpha_{k-1,k})s_k\boldsymbol{Q}_\theta\left[\boldsymbol{z}(\theta_1)\boldsymbol{x}(\alpha_{12}).....\boldsymbol{z}(\theta_{k-1})\boldsymbol{x}(\alpha_{k-1,k})\right]^T \tag{36}$$

$$\boldsymbol{P}_{\alpha_k} = \boldsymbol{z}(\theta_1)\boldsymbol{x}(\alpha_{12}).....\boldsymbol{x}(\alpha_{k-1,k})\boldsymbol{z}(\theta_k)a_{k,k+1}\boldsymbol{Q}_\alpha\left[\boldsymbol{z}(\theta_1)\boldsymbol{x}(\alpha_{12}).....\boldsymbol{x}(\alpha_{k-1,k})\boldsymbol{z}(\theta_k)\right]^T \tag{37}$$

4. Conclusions

The paper presents the methodology of obtaining the closure equation in dual form based on the standard form of the closure equation for a spatial kinematical chain proposed by Hartenberg and Denavit. The identification of the real part and of the coefficients of dual part leads to two matrix equations and finding the solutions of these equations depends on the ratio between the number of translational and the number of rotational pairs. Thus, maximum three translational pairs may exist in the structure of the chain and in this situation, the two systems can be solved independently: first, the one that allows finding the rotations and next, the one that help obtaining the translations. If the number of translational pairs is less than three, the two systems must be considered altogether as a single system that gives simultaneously the solutions both for rotations and translations. By applying certain matrix properties for the matrices occurring in the structure of the dual part of the closure equation, the form of this equation is much simplified, the calculus volume is reduced significantly and an algorithm for finding the equations required for the positional analysis of a spatial mechanism can be formulated.

5. References

[1] Clifford W K 1873 Preliminary sketch of biquaternions *Proc. Lond. Math. Soc.* **4**(64) 381–395

[2] Ball R S 1900 *Theory of Screws* (Cambridge University Press)

[3] Dimentberg F M 1968 *The screw calculus and its applications* Tech. Rep. AD 680993, (Clearinghouse for Federal and Scientific Technical Information VA USA)

[4] Kotelnikov A P 1950 Theory of Vectors and Complex Numbers, in *Several Applications of Lobachevskii's Ideas in Mechanics and Physics, Coll. Papers* (Gostekhizdat, Moscow) 7-47

[5] Yang A T and Freudenstein F 1964 Application of dual number quaternions algebra to the analysis of spatial mechanisms *ASME J. Eng. Ind.* **86** 300–308

[6] Hartenberg R S and Denavit J 1965 *Kinematic synthesis of linkages* (McGraw-Hill series in mechanical engineering, New York: McGraw-Hill)

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